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Matrix difference equations and a nested Bethe ansatz

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Abstract. The system of $SU(N)$- and $U(N)$-matrix difference equations are solved by means of a nested version of a generalized Bethe ansatz, also the called 'off shell' Bethe ansatz (Babujian H M 1990 Correlation functions in WZNW model as a Bethe wavefunction for the Gaudin magnets *Proc. XXIVth Int. Symp. (Ahrenshoop, Zeuthen)*). To solve this new Bethe ansatz in the algebraic language analogous to the conventional case, a new type of monodromy matrices is introduced. They fulfil a new type of Yang–Baxter equations which simplify the proofs. Using a similar approach as for the conventional nested Bethe ansatz the problem is solved iteratively. The vanishing of the 'unwanted terms' of the first level ansatz is equivalent to a set of second level difference equations. The solutions are obtained as sums over 'off-shell' Bethe vectors. These sums are 'Jackson-type integrals'. The highest weight property of the solutions is proved. The solutions are calculated explicitly for several examples of $SU(N)$- and $U(N)$-representations.

1. Introduction

Difference equations play a role in various contexts of mathematical physics (see, for example, [2] and references therein). We are interested in the application to the form factor program in the exact integrable $(1+1)$-dimensional field theory, which was formulated in 1978 by one of the authors (MK) and Weisz [3]. Form factors are matrix elements of local operators $\mathcal{O}(x)$

$$F(\mathrm{i}\pi-\theta)=\langle p'|\mathcal{O}(0)|p\rangle$$

where $p'p=m^2\cosh\theta$. Difference equations for these functions are obtained by Watson's equations [4]

$$F(\theta)=S(\theta)F(-\theta) \qquad F(\mathrm{i}\pi-\theta)=F(\mathrm{i}\pi+\theta) \tag{1.1}$$

where S is the S-matrix. For several models these equations have been solved in [3] and many later publications (see, for example, [5, 6] and references therein). Generalized form factors are matrix elements for many-particle states. For generalized form factors Watson's equations lead typically to matrix difference equations, which can be solved by a generalized Bethe ansatz, also called 'off-shell Bethe ansatz'. The conventional Bethe ansatz introduced by Bethe [7] is used to solve eigenvalue problems. The algebraic formulation, which is also used in this article, was worked out by Faddeev and coworkers (see, for example,

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[8]). The 'off-shell Bethe ansatz' was introduced by one of the authors (HB) to solve the Knizhnik–Zamolodchikov equations which are differential equations. In [9] a variant of this technique was formulated to solve matrix difference equations of the form

$$f(x_1,\ldots,x_i+2,\ldots,x_n) = Q(x_1,\ldots,x_n,;i)f(x_1,\ldots,x_i,\ldots,x_n) \qquad (i=1,\ldots,n)$$

where $f(\underline{x})$ is a vector-valued function and the $Q(\underline{x},i)$ are matrix-valued functions to be specified later. (For further applications of this technique see also [2, 11].) For higher rank internal symmetry groups the nested version of this Bethe ansatz has to be applied. The nested Bethe ansatz used to solve eigenvalue problems was introduced by Yang [10] and further developed by Sutherland [12] (see also [13] for the algebraic formulation and for new applications [11]). The very interesting generalization of this technique, which is applicable to difference equations, is developed in this article for the $SU(N)$ symmetry group. This generalization demonstrates the power of the Bethe ansatz even more beautifully than the conventional applications. In addition we solve difference equations for the $U(N)$ case. It turns out that this problem is much more involved because of the more complicated Bethe ansatz pseudo-groundstate. However, we need these solution to solve the form factor problem for the $SU(N)$ chiral Gross–Neveu model in [17]. Other methods to solve matrix difference equations have been discussed in [5, 14–16].

The article is organized as follows. In section 2 we recall some well known results concerning the $SU(N)$ and $U(N)$ **R**-matrices, the monodromy matrix and some commutation rules. In section 3 we introduce the nested generalized Bethe ansatz to solve a system of matrix difference equations and present the solutions in terms of 'Jackson-type integrals'. The proof of the main theorem avoids the decomposition of the monodromy matrix, as used, for example, in [9]. Instead we introduce a new type of monodromy matrix fulfilling a new type of Yang–Baxter relation and which is adapted to the difference problem. In particular this yields an essential simplification of the proof of the main theorem. In section 4 we prove the highest weight property of the solutions and calculate the weights. Section 5 contains some examples of solutions of the matrix difference equations. As already mentioned compared to the $SU(N)$ case the $U(N)$ case is much more involved. Therefore we separated the treatment of both cases in each section, such that it is possible to skip the $U(N)$ parts but to read and to understand only the $SU(N)$ parts.

2. The *R*-matrix

We consider the vector representations of the spectral parameter depending **R**-matrices as rational solutions of the Yang–Baxter equations.

2.1. The SU(N)-case

Let $V^{1\ldots n}$ be the tensor product space

$$V^{1\ldots n} = V_1 \otimes \cdots \otimes V_n \tag{2.1}$$

where the vector spaces $V_i \cong \mathbb{C}^N$, $(i=1,\ldots,n)$ are considered as fundamental (vector) representation spaces of $SU(N)$. It is straightforward to generalize the results of this paper to the case where the V_i are vector spaces for other representations. We denote the canonical basis vectors by

$$|\alpha_1\ldots\alpha_n\rangle \in V^{1\ldots n} \qquad (\alpha_i = 1,\ldots,N). \tag{2.2}$$

A vector $v^{1\ldots n} \in V^{1\ldots n}$ is given in terms of its components by

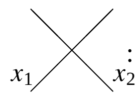
$$v^{1\ldots n} = \sum_{\underline{\alpha}} |\alpha_1 \ldots \alpha_n\rangle v^{\alpha_1,\ldots,\alpha_n}. \tag{2.3}$$

A matrix acting in $V^{1\ldots n}$ by is denoted by

$$A_{1\ldots n}\colon V^{1\ldots n} \to V^{1\ldots n}. \tag{2.4}$$

The $SU(N)$ spectral parameter dependent **R**-matrix [18] acts on the tensor product of two (fundamental) representation spaces of $SU(N)$. It may be written and depicted as

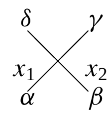
$$R_{12}(x_1 - x_2) = b(x_1 - x_2)\mathbf{1}_{12} + c(x_1 - x_2)P_{12} =$$



$$: V^{12} \to V^{21} \tag{2.5}$$

where P_{12} is the permutation operator. Here and in the following we associate a variable (spectral parameter) $x_i \in \mathbb{C}$ to each space V_i which is graphically represented by a line labelled by x_i (or simply by i). The components of the **R**-matrix are

$$R_{\alpha\beta}^{\delta\gamma}(x_1 - x_2) = \delta_{\alpha\gamma}\delta_{\beta\delta}b(x_1 - x_2) + \delta_{\alpha\delta}\delta_{\beta\gamma}c(x_1 - x_2) = \tag{2.6}$$

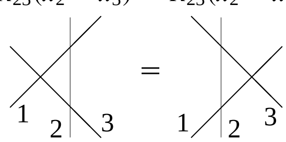


and the functions

$$b(x) = \frac{x}{x - 2/N} \qquad c(x) = \frac{-2/N}{x - 2/N} \tag{2.7}$$

are obtained as the rational solution of the Yang–Baxter equation which reads as and may be depicted as

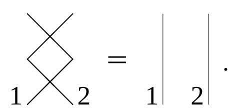
$$R_{12}(x_1 - x_2)R_{13}(x_1 - x_3)R_{23}(x_2 - x_3) = R_{23}(x_2 - x_3)R_{13}(x_1 - x_3)R_{12}(x_1 - x_2)$$



(2.8)

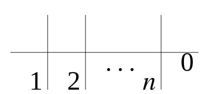
where we have employed the usual notation [10]. The 'unitarity' of the **R**-matrix reads and may depicted as

$$R_{21}(x_2 - x_1)R_{12}(x_1 - x_2) = 1\colon$$



As usual we define the monodromy matrix (with $\underline{x} = x_1, \ldots, x_n$)

$$T_{1\ldots n,0}(\underline{x}, x_0) = R_{10}(x_1 - x_0)R_{20}(x_2 - x_0)\ldots R_{n0}(x_n - x_0) = \tag{2.9}$$



as a matrix acting in the tensor product of the 'quantum space' $V^{1\ldots n}$ and the 'auxiliary space' V_0 (all $V_i \cong \mathbb{C}^N$). The Yang–Baxter algebra relations

$$T_{1\ldots n,a}(\underline{x}, x_a)T_{1\ldots n,b}(\underline{x}, x_b)R_{ab}(x_a - x_b) = R_{ab}(x_a - x_b)T_{1\ldots n,b}(\underline{x}, x_b)T_{1\ldots n,a}(\underline{x}, x_a) \tag{2.10}$$

imply the basic algebraic properties of the sub-matrices with respect to the auxiliary space defined by

$$T_{1\ldots n}{}_{\beta}^{\alpha}(\underline{x}, x) \equiv \begin{pmatrix} A_{1\ldots n}(\underline{x}, x) & B_{1\ldots n\,\beta}(\underline{x}, x) \\ C_{1\ldots n}{}^{\alpha}(\underline{x}, x) & D_{1\ldots n}{}_{\beta}^{\alpha}(\underline{x}, x) \end{pmatrix}. \tag{2.11}$$

The indices α, β on the left-hand side run from 1 to N and on the right-hand side from 2 to N. The commutation rules which we will need later are

$$B_{1\ldots n\alpha}(\underline{x}, x')B_{1\ldots n\beta}(\underline{x}, x) = B_{1\ldots n\beta'}(\underline{x}, x)B_{1\ldots n\alpha'}(\underline{x}, x')R^{\alpha'\beta'}_{\beta\alpha}(x - x') \tag{2.12}$$

$$A_{1\ldots n}(\underline{x}, x')B_{1\ldots n\beta}(\underline{x}, x) = \frac{1}{b(x'-x)}B_{1\ldots n\beta}(\underline{x}, x)A_{1\ldots n}(\underline{x}, x') - \frac{c(x'-x)}{b(x'-x)}B_{1\ldots n\beta}(\underline{x}, x')A_{1\ldots n}(\underline{x}, x) \tag{2.13}$$

and

$$D_{1\ldots n}{}_{\gamma}^{\gamma'}(\underline{x}, x')B_{1\ldots n\beta}(\underline{x}, x) = \frac{1}{b(x-x')}B_{1\ldots n\beta'}(\underline{x}, x)D_{1\ldots n}{}_{\gamma''}^{\gamma'}(\underline{x}, x')R^{\gamma''\beta'}_{\beta\gamma}(x - x') - \frac{c(x-x')}{b(x-x')}B_{1\ldots n\gamma}(\underline{x}, x')D_{1\ldots n}{}_{\beta}^{\gamma'}(\underline{x}, x). \tag{2.14}$$

2.2. The U(N)-case

Let $V^{1\ldots n}$ be the tensor product space

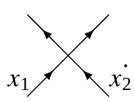
$$V^{1\ldots n} = V_1 \otimes \cdots \otimes V_n \qquad V_i = (V \oplus \bar{V})_i. \tag{2.15}$$

Here $V \cong \mathbb{C}^{\mathbb{N}}$ is considered as a fundamental representation space of $U(N)$ and $\bar{V} \cong \mathbb{C}^{\mathbb{N}}$ as the conjugate representation space. The vectors of V are sometimes called 'particles' with positive $U(1)$ charge and those of $\bar{V}$ 'antiparticles' with negative $U(1)$ charge. We will also use instead of $1, 2, \ldots$ the indices a, b, etc, to refer to the spaces V_a, V_b, etc. It is straightforward to generalize the results of this paper to the case where the V_i are vector spaces for other representations. We denote the canonical basis vectors by

$$|\alpha_1 \ldots \alpha_n\rangle \in V^{1\ldots n} \qquad (\alpha_i = (1, +), \ldots, (N, +), (1, -), \ldots, (N, -)). \tag{2.16}$$

We will later also use the simpler notation $(\alpha, +) \equiv \alpha$ for particles and $(\alpha, -) \equiv \bar{\alpha}$ for antiparticles. A vector in $V^{1\ldots n}$ is denoted by $v^{1\ldots n}$ and a matrix acting in $V^{1\ldots n}$ by $A_{1\ldots n} \in \mathrm{End}(V^{1\ldots n})$.

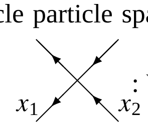
The $U(N)$ spectral parameter depending **R**-matrix $R_{12}(x_1 - x_2)$ [18] acting on the tensor product of two particle spaces $V^{12} = V \otimes V$ coincides with that of $SU(N)$ used in section 2.1 given by (2.5). Here it will be depicted as



. (2.17)

Here and in the following we associate a variable (spectral parameter) $x_i \in \mathbb{C}$ to each space V_i which is graphically represented by a line labelled by x_i (or simply by i). For the $U(N)$-case in addition an arrow on the lines denotes the $U(1)$-charge flow. (If we do not want to specify the direction of charge flow, we draw no arrow.) For $U(N)$ there is in addition the **R**-matrix acting on the tensor product of the antiparticle particle spaces $V^{\bar{1}2} = \bar{V} \otimes V$

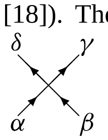
$$R_{\bar{1}2}(x_1 - x_2) = \mathbf{1}_{\bar{1}2} + d(x_1 - x_2)K_{\bar{1}2} =$$



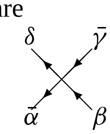
$: V^{\bar{1}2} \to V^{2\bar{1}}$ (2.18)

where $K_{\bar{1}2}$ is the annihilation–creation matrix. There is no particle–antiparticle 'backward scattering' (see, for example, [18]). The components of the **R**-matrices are

$$R^{\delta\gamma}_{\alpha\beta} = \delta_{\alpha\gamma}\delta_{\beta\delta}b + \delta_{\alpha\delta}\delta_{\beta\gamma}c =$$



$$R^{\delta\bar{\gamma}}_{\bar{\alpha}\beta} = \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\beta}\delta_{\delta\gamma}d =$$



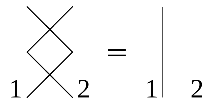
(2.19)

with $\alpha = (\alpha, +)$ etc, and $\bar{\alpha} = (\alpha, -)$ etc. The functions

$$b(x) = \frac{x}{x - 2/N} \qquad c(x) = \frac{-2/N}{x - 2/N} \qquad d(x) = \frac{2/N}{x - 1} \tag{2.20}$$

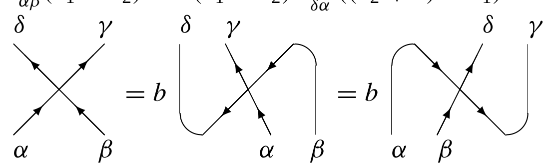
belong to the rational solution of the Yang–Baxter equations (2.8) [18], which holds here for all possible charge flows. The matrices $R_{1\bar{2}}$ and $R_{\bar{1}\bar{2}}$ have the same matrix elements as $R_{\bar{1}2}$ and R_{12}, respectively. The inversion ('unitarity') relation of the **R**-matrix reads as and may be depicted as

$$R_{21}(x_2 - x_1)R_{12}(x_1 - x_2) = 1: \tag{2.21}$$

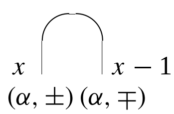


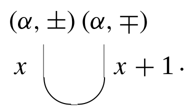
again for all possible charge flows. A further property of the $U(N)$ **R**-matrices is crossing and may be written and depicted as

$$R^{\delta\gamma}_{\alpha\beta}(x_1 - x_2) = b(x_1 - x_2)R^{\gamma\bar{\beta}}_{\bar{\delta}\alpha}((x_2 + 1) - x_1) = b(x_1 - x_2)R^{\bar{\alpha}\delta}_{\beta\bar{\gamma}}(x_2 - (x_1 - 1))$$



where again we have used the notation $\alpha = (\alpha, +)$ etc, and $\bar{\alpha} = (\alpha, -)$ etc. We have introduced the graphical rule that a line changing the 'time direction' also interchanges particles and antiparticles and changes $x \to x \pm 1$ as follows



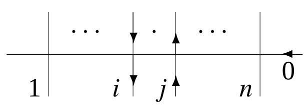


(2.22)

In a similar way as in the above we introduce a monodromy matrix

$$T_{1\ldots n,0}(\underline{x}; x_0) = R_{10}(x_1 - x_0)\ldots R_{n0}(x_n - x_0) \tag{2.23}$$

$$=$$



as a matrix acting in the tensor product of the 'quantum space' $V^{1\ldots n}$ and the 'auxiliary space' $V_0 \cong V \cong \mathbb{C}^N$ as a particle space. Since there is no 'charge reflection' the positions of the particles and the antiparticles will not change under the application of $T_{1\ldots n,0}$ to a state in $V^{1\ldots n}$. (The construction of the Bethe states will not be symmetric with respect to particles and antiparticles, because we only use the monodromy matrix with particles for the auxiliary space.)

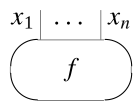
The Yang–Baxter algebra relations (2.10) also hold for the $U(N)$-monodromy matrix and the resulting commutation rules (2.12)–(2.14) of the A, B, C, D matrices are the same as for the $SU(N)$ case of section 2.1.

3. The matrix difference equation and 'generalized Bethe vectors'

In this section we formulate the matrix difference equations for vector-valued functions and solve them by means of the nested version of the Bethe ansatz, a variant of the Bethe ansatz also called 'off-shell Bethe ansatz'. We will call the solutions 'generalized Bethe vectors'. The conventional Bethe ansatz is used to solve eigenvalue problems and leads to the 'Bethe ansatz equations'. This system of equations can usually be solved only for particular cases. Here the Bethe ansatz leads to some simple functional equations which can be solved easily and the solutions of the difference equations are given in terms of infinite sums called 'Jackson-type integrals'.

3.1. The SU(N)-case

Let

$$f^{1\ldots n}(\underline{x}) =$$ 

$$\in V^{1\ldots n}$$

be a vector-valued function of $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{C}^n$ with values in $V^{1\ldots n}$. The components of this vector are denoted by

$$f^{\alpha_1\ldots\alpha_n}(\underline{x}) \qquad (1 \leqslant \alpha_i \leqslant N).$$

Conditions 3.1. The following symmetry and periodicity conditions of the vector-valued function $f^{1\ldots n}(\underline{x})$ are supposed to be valid:

(i) The symmetry property under the exchange of two neighbouring spaces V_i and V_j and the variables x_i and x_j, at the same time, is given by

$$f^{\ldots ji\ldots}(\ldots, x_j, x_i, \ldots) = R_{ij}(x_i - x_j) f^{\ldots ij\ldots}(\ldots, x_i, x_j, \ldots). \tag{3.1}$$

(ii) The *system of matrix difference equations* holds

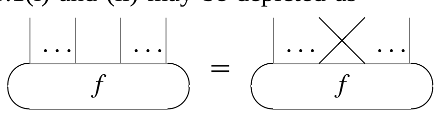
$$f^{1\ldots n}(\ldots, x_i + 2, \ldots) = Q_{1\ldots n}(\underline{x}; i) f^{1\ldots n}(\ldots, x_i, \ldots) \qquad (i = 1, \ldots, n) \tag{3.2}$$

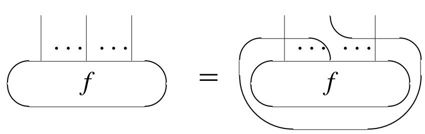
where the matrices $Q_{1\ldots n}(\underline{x}; i) \in \mathrm{End}(V^{1\ldots n})$ are defined by

$$Q_{1\ldots n}(\underline{x}; i) = R_{i+1i}(x_{i+1} - x_i') \ldots R_{ni}(x_n - x_i') R_{1i}(x_1 - x_i) \ldots R_{i-1i}(x_{i-1} - x_i) \tag{3.3}$$

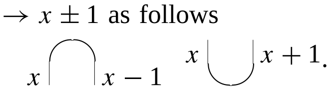
with $x_i' = x_i + 2$.

The Yang–Baxter equations for the **R**-matrix guarantee that these conditions are compatible. The shift of two in (3.2) could be replaced by an arbitrary κ. For the application to the form factor problem, however, it is fixed to two because of crossing symmetry. Conditions 3.1(i) and (ii) may be depicted as

(i) 

(ii) 

with the graphical rule that a line changing the 'time direction' changes the spectral parameters $x \to x \pm 1$ as follows



The $Q_{1\ldots n}(\underline{x}; i)$ fulfil the commutation rules

$$\begin{aligned} &Q_{1\ldots n}(\ldots x_i \ldots x_j + 2 \ldots; i) Q_{1\ldots n}(\ldots x_i \ldots x_j \ldots; j) \\ &\qquad = Q_{1\ldots n}(\ldots x_i + 2 \ldots x_j \ldots; j) Q_{1\ldots n}(\ldots x_i \ldots x_j \ldots; i). \end{aligned} \tag{3.4}$$

The following proposition is obvious

Proposition 3.2. Let the vector-valued function $f^{1\ldots n}(\underline{x}) \in V^{1\ldots n}$ fulfil condition 3.1(i), then conditions 3.1(ii) for all i $= 1, \ldots, n$ are equivalent to each other and also equivalent to the following periodicity property under cyclic permutation of the spaces and the variables

$$f^{12\ldots n}(x_1, x_2, \ldots, x_n + 2) = f^{n1\ldots n-1}(x_n, x_1, \ldots, x_{n-1}). \tag{3.5}$$

Remark 3.3. The equations (3.1) and (3.5) imply Watson's (1.1) equations for the form factors [17].

For later convenience we write the matrices

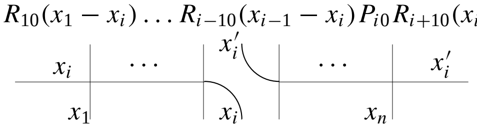
$$Q_{1\ldots n}(\underline{x}; i) = \mathrm{tr}_0 T^{Q}_{1\ldots n,0}(\underline{x}; i) \tag{3.6}$$

as the trace of a new type of monodromy matrices where to the horizontal line two different spectral parameters are associated, namely one to the right-hand side and the other one to the left-hand side. However, both are related to a spectral parameter of one of the vertical lines. This new monodromy matrix is given by the following

Definition 3.4. For $i = 1, \ldots, n$

$$T^{Q}_{1\ldots n,0}(\underline{x}; i) = R_{10}(x_1 - x_i) \ldots R_{i-10}(x_{i-1} - x_i) P_{i0} R_{i+10}(x_{i+1} - x'_i) \ldots R_{n0}(x_n - x'_i) \tag{3.7}$$

$=$



with $x'_i = x_i + 2$.

Note that for $i = n$ one has simply $T^{Q}_{1\ldots n,0}(\underline{x}; n) = T_{1\ldots n,0}(\underline{x}, x_n)$ since $R(0)$ is the permutation operator P.

The new type of monodromy matrix fulfils a new type of Yang–Baxter relation. Instead of (2.10) we have for $i = 1, \ldots, n$

$$T^{Q}_{1\ldots n,0}(\underline{x}; i) T_{1\ldots n,b}(\underline{x}, u) R_{ab}(x'_i - u) = R_{ab}(x_i - u) T_{1\ldots n,b}(\underline{x}', u) T^{Q}_{1\ldots n,a}(\underline{x}; i) \tag{3.8}$$

with $\underline{x}' = x_1, \ldots, x'_i, \ldots, x_n$ and $x'_i = x_i + 2$. This relation follows from the Yang–Baxter equation for the **R**-matrix and the obvious relation for the permutations operator P

$$P_{ia} R_{ib}(x_i - u) R_{ab}(x'_i - u) = R_{ab}(x_i - u) R_{ib}(x'_i - u) P_{ia}.$$

Correspondingly to (2.11) we introduce (suppressing the indices $1 \ldots n$)

$$T^{Q\alpha}{}_{\beta}(\underline{x}; i) \equiv \begin{pmatrix} A^{Q}(\underline{x}; i) & B^{Q}{}_{\beta}(\underline{x}; i) \\ C^{Q\alpha}(\underline{x}; i) & D^{Q\alpha}{}_{\beta}(\underline{x}; i) \end{pmatrix} \tag{3.9}$$

with the commutation rules with respect to the usual A, B, C, D

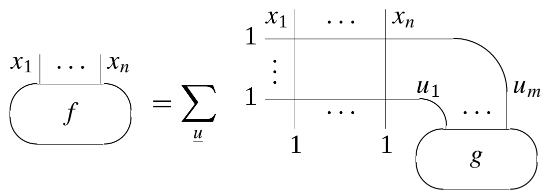
$$A^{Q}(\underline{x}; i) B_b(\underline{x}, u) = \frac{1}{b(x'_i - u)} B_b(\underline{x}', u) A^{Q}(\underline{x}; i) - \frac{c(x'_i - u)}{b(x'_i - u)} B^{Q}{}_b(\underline{x}; i) A(\underline{x}, u) \tag{3.10}$$

$$\begin{aligned} D^{Q}{}_a(\underline{x}; i) B_b(\underline{x}, u) = {} & \frac{1}{b(u - x_i)} B_b(\underline{x}', u) D^{Q}{}_a(\underline{x}; i) R_{ba}(u - x'_i) \\ & - \frac{c(u - x_i)}{b(u - x_i)} B^{Q}{}_b(\underline{x}; i) D_a(\underline{x}, u) P_{ab}. \end{aligned} \tag{3.11}$$

The system of difference equations (3.2) can be solved by means of a generalized ('off-shell') nested Bethe ansatz. The first level is given by the following.

Bethe ansatz 3.5.

$$f^{1\dots n}(\underline{x}) = \sum_{\underline{u}} B_{1\dots n\beta_m}(\underline{x}, u_m)\dots B_{1\dots n,\beta_1}(\underline{x}, u_1)\Omega^{1\dots n} g^{\beta_1\dots\beta_m}(\underline{x}, \underline{u}) \tag{3.12}$$



where summation over $\beta_1, \dots, \beta_m$ is assumed and $\Omega^{1\dots n} \in V^{1\dots n}$ is the reference state defined by $C_{1\dots n}{}^{\beta}\Omega^{1\dots n} = 0$ for $1 < \beta \leqslant N$. The summation over $\underline{u}$ is specified by

$$\underline{u} = (u_1, \dots, u_m) = (\tilde{u}_1 - 2l_1, \dots, \tilde{u}_m - 2l_m) \qquad l_i \in \mathbb{Z} \tag{3.13}$$

where the $\tilde{u}_i$ are arbitrary constants.

The reference state is

$$\Omega^{1\dots n} = |1\dots 1\rangle \tag{3.14}$$

a basis vector with components $\Omega^{\alpha_1\dots\alpha_n} = \prod_{i=1}^{n} \delta_{\alpha_i 1}$. It is an eigenstate of $A_{1\dots n}$ and $D_{1\dots n}$

$$A_{1\dots n}(\underline{x}, u)\Omega^{1\dots n} = \Omega^{1\dots n} \qquad D_{1\dots n}{}^{\alpha}_{\beta}(\underline{x}, u)\Omega^{1\dots n} = \Omega^{1\dots n}\delta_{\alpha\beta}\prod_{i=1}^{n} b(x_i - u). \tag{3.15}$$

The sums (3.12) are also called 'Jackson-type integrals' (see, for example, [9] and references therein). Note that the summations over β_i run only over $1 < \beta_i \leqslant N$. Therefore the $g^{\beta_1\dots\beta_m}$ are the components of a vector $g^{b_1\dots b_m}$ in the tensor product of smaller spaces

$$g^{b_1\dots b_m} \in V^{(1)b_1\dots b_m} = V_1^{(1)} \otimes \cdots \otimes V_m^{(1)} \tag{3.16}$$

with

$$V_i^{(1)} = \{|\beta\rangle; 1 < \beta \leqslant N\} \cong \mathbb{C}^{N-1}. \tag{3.17}$$

We define a new vector-valued function $f^{(1)^{b_1\dots b_m}}(\underline{u}) \in V^{(1)^{b_1\dots b_m}}$ by

Definition 3.6. Let the vector valued function $g^{b_1\dots b_m}(\underline{u}) \in V^{(1)^{b_1\dots b_m}}$ be given by

$$g^{b_1\dots b_m}(\underline{x}, \underline{u}) = \prod_{i=1}^{n}\prod_{j=1}^{m}\psi(x_i - u_j)\prod_{1\leqslant i<j\leqslant m}\tau(u_i - u_j) f^{(1)^{b_1\dots b_m}}(\underline{u}) \tag{3.18}$$

where the functions $\psi(x)$ and $\tau(x)$ fulfil the functional equations

$$b(x)\psi(x) = \psi(x-2) \qquad \frac{\tau(x)}{b(x)} = \frac{\tau(x-2)}{b(2-x)}. \tag{3.19}$$

Using the definition of $b(x)$ (2.7) we get the solutions of (3.19)

$$\psi(x) = \frac{\Gamma(1 - N^{-1} + \frac{1}{2}x)}{\Gamma(1 + \frac{1}{2}x)} \qquad \tau(x) = \frac{x\Gamma(N^{-1} + \frac{1}{2}x)}{\Gamma(1 - N^{-1} + \frac{1}{2}x)} \tag{3.20}$$

where the general solutions are obtained by multiplication with arbitrary periodic functions with period two. Just as $g^{1\dots m}(\underline{x}, \underline{u})$ also the vector-valued function $f^{(1)^{b_1\dots b_m}}(\underline{u})$ is an element of the tensor product of the smaller spaces $V_i^{(1)} \cong \mathbb{C}^{N-1}$

$$f^{(1)^{b_1\dots b_m}}(\underline{u}) \in V^{(1)^{b_1\dots b_m}}.$$

We say $f^{(1)b_1\ldots b_m}(\underline{u})$ fulfils conditions 3.1(i)$^{(1)}$ and (ii)$^{(1)}$ if (3.1) and (3.2) hold in this space, which means that everywhere in (3.1) and (3.3) the matrix **R** has to be replaced by

$$R^{(1)} \equiv R \text{ restricted to } V^{(1)} \otimes V^{(1)} \tag{3.21}$$

(cf (3.17)). We are now in a position to formulate the main theorem of this paper.

Theorem 3.7. Let the vector-valued function $f^{1\ldots n}(\underline{x})$ be given by the Bethe ansatz 3.5 and let $g^{b_1\ldots b_m}(\underline{x}, \underline{u})$ be of the form of definition 3.6. If in addition the vector-valued function $f^{(1)b_1\ldots b_m}(\underline{u}) \in V^{(1)b_1\ldots b_m}$ fulfils conditions 3.1(i)$^{(1)}$ and (ii)$^{(1)}$, then $f^{1\ldots n}(\underline{x}) \in V^{1\ldots n}$ fulfils the conditions 3.1(i) and (ii), i.e. $f^{1\ldots n}(\underline{x})$ is a solution of the set of difference (3.2).

Remark 3.8. For $SU(2)$ (see, for example, [9]) the problem is already solved by (3.18) since then $f^{(1)}$ is a constant.

Proof. Condition 3.1(i) follows directly from the definition and the normalization of the **R**-matrix (2.5)

$$R_{ij}(x_i - x_j)\Omega^{\ldots ij\ldots} = \Omega^{\ldots ij\ldots}$$

the symmetry of $g^{1\ldots m}(\underline{x}, \underline{u})$ given by (3.18) under the exchange of $x_1, \ldots, x_n$ and

$$B_{\ldots ji\ldots,\beta}(\ldots x_j, x_i \ldots, u)R_{ij}(x_i - x_j) = R_{ij}(x_i - x_j)B_{\ldots ij\ldots,\beta}(\ldots x_i, x_j \ldots, u)$$

which is a consequence of the Yang–Baxter relations for the **R**-matrix.

Because of proposition 3.2 it is sufficient to prove condition 3.1(ii) only for $i = n$

$$Q(\underline{x}; n)f(\underline{x}) = \mathrm{tr}_a T_a^Q(\underline{x}; n)f(\underline{x}) = f(\underline{x}') \qquad (\underline{x}' = x_1, \ldots, x_n' = x_n + 2)$$

where the indices $1\ldots n$ have been suppressed. For the first step we apply a technique quite analogous to that used for the conventional algebraic Bethe ansatz which solves eigenvalue problems. We apply the trace of $T_a^Q(\underline{x}; n)$ to the vector $f(\underline{x})$ as given by (3.12) and push $A^Q(\underline{x}; n)$ and $D_a^Q(\underline{x}; n)$ through all the Bs using the commutation rules (3.10) and (3.11). Again with $\underline{x}' = x_1, \ldots, x_n' = x_n + 2$ we obtain

$$\begin{aligned} &A^Q(\underline{x}; n)B_{b_m}(\underline{x}, u_m)\ldots B_{b_1}(\underline{x}, u_1) \\ &\qquad = B_{b_m}(\underline{x}', u_m)\ldots B_{b_1}(\underline{x}', u_1)\prod_{j=1}^m \frac{1}{b(x_n' - u_j)} A^Q(\underline{x}; n) + \mathrm{uw}_A \end{aligned}$$

$$\begin{aligned} &D_a^Q(\underline{x}; n)B_{b_m}(\underline{x}, u_m)\ldots B_{b_1}(\underline{x}, u_1) \\ &\qquad = B_{b_m}(\underline{x}', u_m)\ldots B_{b_1}(\underline{x}', u_1)\prod_{j=1}^m \frac{1}{b(u_j - x_n)} D_a^Q(\underline{x}; n)R_{b_1 a}^{(1)}(u_1 - x_n')\ldots \\ &\qquad\quad \ldots R_{b_m a}^{(1)}(u_m - x_n') + \mathrm{uw}_{D_a} \end{aligned}$$

where $R^{(1)}$ is defined by (3.21). The 'wanted terms' written explicitly originate from the first term in the commutations rules (3.10) and (3.11); all other contributions yield the 'unwanted terms'. If we insert these equations into the representation (3.12) of $f(\underline{x})$ we find that the wanted contribution from A^Q already gives the desired result. The wanted contribution from D^Q applied to Ω gives zero. The unwanted contributions can be written as a difference which vanishes after summation over the us. These three facts can be seen as follows. We have

$$A^Q(\underline{x}; n)\Omega = \Omega \qquad D_a^Q(\underline{x}; n)\Omega = 0$$

which follow from (3.15) since $T^Q(\underline{x}; n) = T(\underline{x}, x_n)$ and $b(0) = 0$. The defining relation of $\psi(x)$ (3.19) implies that the wanted term from A yields $f(\underline{x}')$. The commutation relations (3.10), (3.11), (2.13) and (2.14) imply that the unwanted terms are proportional to a product of B-operators, where exactly one $B_{b_j}(\underline{x}, u_j)$ is replaced by $B^Q_{b_j}(\underline{x}; n)$. Because of the commutation relations of the Bs (2.12) and the symmetry property given by condition 3.1(i)$^{(1)}$ of $g^{1\ldots m}(\underline{x}, \underline{u})$ it is sufficient to consider only the unwanted terms for $j = m$ denoted by uw^m_A and uw^m_D. They come from the second term in (3.10) if $A^Q(\underline{x}; n)$ is commuted with $B_{b_m}(\underline{x}, u_m)$ and then the resulting $A(\underline{x}, u_m)$ pushed through the other Bs taking only the first terms in (2.13) into account and correspondingly for $D^Q_a(\underline{x}; u_m)$.

$$\mathrm{uw}^m_A = -\frac{c(x'_n - u_m)}{b(x'_n - u_m)} B^Q_{b_m}(\underline{x}; m) \ldots B_{b_1}(\underline{x}, u_1) \prod_{j<m} \frac{1}{b(u_m - u_j)} A(\underline{x}, u_m)$$

$$\mathrm{uw}^m_{D_a} = -\frac{c(u_m - x_n)}{b(u_m - x_n)} B^Q_{b_m}(\underline{x}; m) \ldots B_{b_1}(\underline{x}, u_1) \prod_{j<m} \frac{1}{b(u_j - u_m)} D_a(\underline{x}, u_m) T^{Q(1)}_{b_1\ldots b_m,a}(\underline{u}; m)$$

where $T^{Q(1)}$ is the new type of monodromy matrix

$$T^{Q(1)}_{b_1\ldots b_m,a}(\underline{u}; m) = R^{(1)}_{b_1 a}(u_1 - u_m) \ldots R^{(1)}_{b_{m-1} a}(u_{m-1} - u_m) P_{b_m a} \qquad (3.22)$$

analogous to (3.6) whose trace over the auxiliary space $V^{(1)}_a$ yields the shift operator $Q^{(1)}(\underline{u}; m)$. With $D_a(\underline{x}, u_m)\Omega = \mathbf{1}_a \prod_{i=1}^n b(x_i - u_m)\Omega$ (see (3.15)), by the assumption

$$Q^{(1)}(\underline{u}; m) f^{(1)}(\underline{u}) = f^{(1)}(\underline{u}') \qquad (\underline{u}' = u_1, \ldots, u'_m = u_m + 2)$$

and the defining relations (3.19) of $\psi(x)$ and $\tau(x)$, we obtain

$$\mathrm{tr}_a \mathrm{uw}^m_{D_a}(\underline{u}) \Omega g(\underline{x}, \underline{u}) = -\mathrm{uw}^m_A(\underline{u}') \Omega g(\underline{x}, \underline{u}')$$

where $c(-x)/b(-x) = -c(x)/b(x)$ has been used. Therefore the sum of all unwanted terms yield a difference analogous of a total differential which vanishes after summation over the us.

Iterating theorem 3.7 we obtain the nested generalized Bethe ansatz with levels $k = 1, \ldots, N-1$. The ansatz of level k reads

$$f^{(k-1)^{1\ldots n_{k-1}}}(\underline{x}^{(k-1)}) = \sum_{\underline{x}^{(k)}} B^{(k-1)}_{1\ldots n_{k-1}\beta_{n_k}}(\underline{x}^{(k-1)}, x^{(k)}_{n_k}) \ldots$$
$$\ldots B^{(k-1)}_{1\ldots n_{k-1}\beta_1}(\underline{x}^{(k-1)}, x^{(k)}_1) \Omega^{(k-1)^{1\ldots n_{k-1}}} g^{(k-1)^{\beta_1\ldots\beta_{n_k}}}(\underline{x}^{(k-1)}, \underline{x}^{(k)}). \qquad (3.23)$$

The functions $f^{(k)}$ and $g^{(k)}$ are vectors with

$$f^{(k)^{1\ldots n_k}}, g^{(k-1)^{1\ldots n_k}} \in V^{(k)^{1\ldots n_k}} = V^{(k)}_1 \otimes \cdots \otimes V^{(k)}_{n_k} \qquad (V^{(k)}_i \cong \mathbb{C}^{N-k}).$$

The basis vectors of these spaces are $|\alpha_1 \ldots \alpha_{n_k}\rangle^{(k)} \in V^{(k)^{1\ldots n_k}}$ and $k < \alpha_i \leqslant N$.

Analogously to definition 3.6 we write

$$g^{(k-1)^{1\ldots n_k}}(\underline{x}^{(k-1)}, \underline{x}^{(k)}) = \prod_{i=1}^{n_{k-1}} \prod_{j=1}^{n_k} \psi(x^{(k-1)}_i - x^{(k)}_j) \prod_{1\leqslant i<j\leqslant n_k} \tau(x^{(k)}_i - x^{(k)}_j) f^{(k)^{1\ldots n_k}}(\underline{x}^{(k)}) \qquad (3.24)$$

where the functions $\psi(x)$ and $\tau(x)$ fulfil the functional equations (3.19) with the solutions (3.20). Then the start of the iteration is given by a $k_{\max} \leqslant N$ with

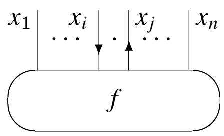
$$f^{(k_{\max}-1)^{1\ldots n_{k_{\max}-1}}} = |k_{\max} \ldots k_{\max}\rangle \qquad \text{and } n_k = 0 \text{ for } k \geqslant k_{\max} \qquad (3.25)$$

which is the reference state of level $k_{\max} - 1$ and trivially fulfils the conditions 3.1.

Corollary 3.9. The system of $SU(N)$ matrix difference equations (3.2) is solved by the nested Bethe ansatz (3.23) with (3.24), (3.25) and $f^{1\ldots n}(\underline{x}) = f^{(0)^{1\ldots n}}(\underline{x})$.

3.2. The U(N)-case

Let again

$$f^{1\dots n}(\underline{x}) =$$ 

$$\in V^{1\dots n}$$

be a vector-valued function of $\underline{x} = x_1, \dots, x_n$ with values in $V^{1\dots n}$ which is now given by (2.15).

Analogously to the $SU(N)$-case in section 3.1 we constrain the function $f^{1\dots n}$ by the conditions 3.1. The $Q_{1\dots n}(\underline{x}; i)$ fulfil the same commutation rules (3.4) as in section 3.1. Also proposition 3.2 and remark 3.3 hold for the $U(N)$ case. We write the **Q**-matrices again as the trace (3.6) of the new type of monodromy matrices analogous to those given by definition 3.4. We use this monodromy matrix if the index i is associated with a particle. If the index i is associated with an antiparticle, we will use an additional monodromy matrix of a new type which yields the inverse of the **Q**-operators.

$$Q^{-1}_{1\dots n}(\underline{x}; i) = \tilde{Q}_{1\dots n}(\underline{x}; i) = \mathrm{tr}_{\bar{0}} \tilde{T}^{Q}_{\bar{0},1\dots n}(\underline{x}; i). \tag{3.26}$$

It is given by the following

Definition 3.10. For $i = 1, \dots, n$

$$\tilde{T}^{Q}_{\bar{0},1\dots n}(\underline{x}; i) = R_{\bar{0}1}(x_i - x_1) \dots R_{\bar{0}i-1}(x_i - x_{i-1}) P_{\bar{0}\bar{i}} R_{\bar{0}i+1}(x'_i - x_{i+1}) \dots R_{\bar{0}n}(x'_i - x_n) \tag{3.27}$$

with the auxiliary space $V_{\bar{0}} = \bar{V}$ and $x'_i = x_i + 2$.

The 'unitarity' of the **R**-matrix (2.21) implies (3.26).

The new type of monodromy matrix $\tilde{T}^{Q}_{\bar{0},1\dots n}$ fulfils a new type of Yang–Baxter relation of the form

$$\tilde{T}^{Q}_{a,1\dots n}(\underline{x}; i) R_{\bar{a}b}(x_i - u) T_{1\dots n,b}(\underline{x}', u) = T_{1\dots n,b}(\underline{x}, u) R_{\bar{a}b}(x'_i - u) \tilde{T}^{Q}_{a,1\dots n}(\underline{x}; i) \tag{3.28}$$

with $\underline{x}' = x_1, \dots, x'_i, \dots, x_n$ and $x'_i = x_i + 2$. In addition to the commutation rules (3.10) and (3.11) for the case that the indices i and a belong to particles, we have the commutation rules for the case that the indices i and a correspond to antiparticles

$$\tilde{A}^{Q}(\underline{x}; i) B_b(\underline{x}', u) = \frac{1 - d^2(x'_i - u)}{1 + d(x_i - u)} B_b(\underline{x}', u) \tilde{A}^{Q}(\underline{x}; i) + d(x'_i - u) \tilde{B}^{Q}_{\bar{b}}(\underline{x}; i) A(\underline{x}', u) \tag{3.29}$$

$$\tilde{D}^{Q}_{\bar{a}}(\underline{x}; i) B_b(\underline{x}', u) = B_b(\underline{x}, u) R_{\bar{a}b}(x'_i - u) \tilde{D}^{Q}_{\bar{a}}(\underline{x}; i) - d(x_i - u) \tilde{B}^{Q}_{\bar{a}}(\underline{x}; i) K_{\bar{a}b} D_b(\underline{x}', u). \tag{3.30}$$

The system of difference equations (3.2) can be solved by means of a generalized ('off-shell') nested Bethe ansatz. The first level is given by the following.

Bethe ansatz 3.11.

$$f^{1\dots n}(\underline{x}) = \sum_{\underline{u}} B_{1\dots n\beta_m}(\underline{x}, u_m) \dots B_{1\dots n\beta_1}(\underline{x}, u_1) g^{1\dots n\beta_1\dots\beta_m}(\underline{x}, \underline{u}) \tag{3.31}$$

where summation over $1 < \beta_1, \ldots, \beta_m \leqslant N$ is assumed. The summation over $\underline{u}$ is specified by

$$\underline{u} = (u_1, \ldots, u_m) = (\tilde{u}_1 - 2l_1, \ldots, \tilde{u}_m - 2l_m) \qquad l_i \in \mathbb{Z} \tag{3.32}$$

where the $\tilde{u}_i$ are arbitrary constants. For a fixed set of indices $\underline{\beta} = \beta_1, \ldots, \beta_m$ $(1 < \beta_i \leqslant N)$ the reference state vector $g^{1\ldots n}\underline{\beta} \in V^{1\ldots n}$ fulfils

$$C_{1\ldots n}{}^{\gamma} g^{1\ldots n\underline{\beta}} = 0 \qquad (1 < \gamma \leqslant N). \tag{3.33}$$

Note that we have denoted here by g, what was called Ωg in section 3.1. This notation is more convenient, since in contrast to the $SU(N)$ case the space of reference states is higher dimensional for $U(N)$. This space V_{ref} of states fulfilling (3.33) is spanned by all basis vectors of the form

$$|\alpha_1, \ldots, \alpha_n\rangle \qquad \alpha_i = \begin{cases} (1, +) & \text{for particles} \\ (\alpha_i, -) \qquad (1 < \alpha_i \leqslant N) & \text{for antiparticles.} \end{cases}$$

The $U(1)$-charge is left invariant by operators like the monodromy matrices $T_{1\ldots n,0}$, $T^{Q}_{1\ldots n,0}$, $\tilde{T}^{Q}_{1\ldots n,0}$ and also by the operations of conditions 3.1. Therefore the space $V^{1\ldots n}$ decomposes into invariant subspaces of fixed charge, i.e. fixed numbers n_-, n_+ $(n_- + n_+ = n)$ of antiparticles and particles, respectively. Moreover because of the symmetry property (i) it is sufficient to consider the even smaller subspaces $V^{I_- I_+}$ where the antiparticles and the particles are sitting at fixed places I_- and I_+, respectively

$$V^{I_- I_+} = V^{I_-} \otimes V^{I_+} \qquad V^{I_-} = \bigotimes_{i \in I_-} \bar{V}_i \qquad V^{I_+} = \bigotimes_{i \in I_+} V_i \tag{3.34}$$

such that

$$V^{1\ldots n} = \bigoplus_{I_- \cup I_+ = \{1,\ldots,n\}} V^{I_- I_+}. \tag{3.35}$$

In the following we consider the case where the antiparticles are sitting at the first n_- places, i.e. $I_- = \{1, \ldots, n_-\}$. Then the space of reference states in the subspace $V^{I_- I_+}$ may be written as

$$V_{ref} \cap V^{I_- I_+} \cong \bar{V}^{(1)}_1 \otimes \cdots \otimes \bar{V}^{(1)}_{n_-} \otimes \Omega^{I_+} \tag{3.36}$$

where the spaces $\bar{V}^{(1)}_i \cong \mathbb{C}^{N-1}$ contain only antiparticle states with $\bar{\alpha} > 1$ and the vector Ω^{I_+} consists of particles with $\alpha = 1$

$$\Omega^{I_+} = |1 \ldots 1\rangle \in V^{I_+}$$

Therefore, as in (3.16) g may also be considered as a vector

$$g^{1\ldots n b_1\ldots b_m} g \in V^{(1)^{I_- b_1\ldots b_m}} \otimes \Omega^{I_+}$$

with

$$V^{(1)^{I_- b_1\ldots b_m}} = \bigotimes_{i \in I_-} \bar{V}^{(1)}_i \otimes V^{(1)}_{b_1} \otimes \cdots \otimes V^{(1)}_{b_m} \tag{3.37}$$

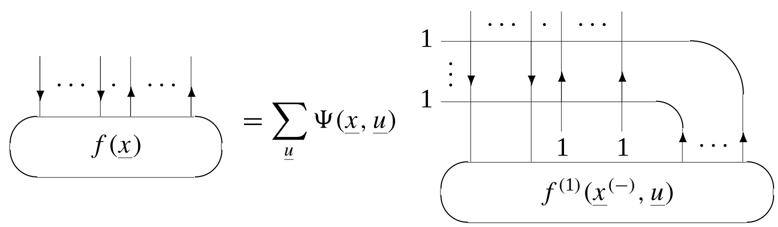
where the $\bar{V}^{(1)}_i \cong \mathbb{C}^{N-1}$ are the smaller antiparticle spaces and the $V^{(1)}_{b_i} \cong \mathbb{C}^{N-1}$ are smaller particle spaces. Again as in section 3.1 we define a new vector-valued function $f^{(1)^{I_- b_1\ldots b_m}} \in V^{(1)^{I_- b_1\ldots b_m}}$ by

Definition 3.12. Let $g^{1\ldots nb_1\ldots b_m}g(\underline{x},\underline{u})$ be given in the subspace with fixed positions of anti-particles I_- and particles I_+ as

$$g^{1\ldots nb_1\ldots b_m}(\underline{x},\underline{u}) = \prod_{i\in I_+}\prod_{j=1}^{m}\psi(x_i-u_j)\prod_{1\leqslant i<j\leqslant m}\tau(u_i-u_j)\, f^{(1)^{I_-b_1\ldots b_m}}(\underline{x}^{(-)},\underline{u})\otimes\Omega^{I_+} \qquad (3.38)$$

where $\underline{x}^{(-)} = \{x_i;\, i\in I_-\}$ are the spectral parameters of the anti-particles only.

The functions $\psi(x)$ and $\tau(x)$ are the same as in section 3.1 given by (3.20). For the case that the antiparticles are sitting at the first n_- places the Bethe ansatz 3.11 may be depicted as



where $\Psi(\underline{x},\underline{u})$ is given by the products of ψs and τs of (3.38). The main theorem of this article is analogous to that of section 3.1.

Theorem 3.13. Let the vector-valued function $f^{1\ldots n}(\underline{x})$ be given by the Bethe ansatz 3.11 where $g^{1\ldots nb_1\ldots b_m}(\underline{x},\underline{u})$ is of the form of definition 3.12. If in addition the vector-valued function $f^{(1)^{I_-b_1\ldots b_m}}(\underline{x}^{(-)},\underline{u}) \in V^{(1)^{I_-b_1\ldots b_m}}$ fulfils conditions 3.1(i)$^{(1)}$ and (ii)$^{(1)}$, then $f^{1\ldots n}(\underline{x}) \in V$ fulfils conditions 3.1(i) and (ii), i.e. $f^{1\ldots n}(\underline{x})$ is a solution of the set of difference (3.2).

Proof. Condition 3.1(i) follows as for the $SU(N)$ case. Again it is sufficient to prove condition 3.1(ii) for $i=n$

$$Q(\underline{x};n)f(\underline{x}) = f(\underline{x}') \qquad (\underline{x}' = x_1,\ldots,x_n' = x_n+2).$$

where the indices $1\ldots n$ have been suppressed. For the case that the index n belongs to a particle the proof is similar to that in section 3.1. The only difference is that here the next level **Q**-matrix is more complicated. Instead of (3.22) we have here

$$T^{Q(1)}_{\underline{b}a}(\underline{x}^{(-)},\underline{u};b_m) = \prod_{i\in I_-}R^{(1)}_{ia}(x_i-u_m)R^{(1)}_{b_1a}(u_1-u_m)\ldots R^{(1)}_{b_{m-1}a}(u_{m-1}-u_m)P_{b_ma}. \qquad (3.39)$$

For the case that the index n belongs to an antiparticle we will prove the inverse of (ii) using $\tilde{T}^{Q}_{\bar{a},1\ldots n}$. We apply $Q^{-1}(\underline{x};n) = \tilde{Q}(\underline{x};n)$ given by the trace of $\tilde{T}^{Q}_{\bar{a},1\ldots n}(\underline{x};n)$ (cf (3.27)) to the vector $f^{1\ldots n}(\underline{x}')$ as given by (3.31) and push $\tilde{A}^{Q}(\underline{x};n)$ and $\tilde{D}^{Q}_{\bar{a}}(\underline{x};n)$ through all the Bs using the commutation rules (3.29) and (3.30). Again with $\underline{x}' = x_1,\ldots,x_n' = x_n+2$ we obtain

$$\begin{aligned}
&\tilde{A}^{Q}(\underline{x};n)B_{b_m}(\underline{x}',u_m)\ldots B_{b_1}(\underline{x}',u_1)\\
&\qquad = B_{b_m}(\underline{x},u_m)\ldots B_{b_1}(\underline{x},u_1)\prod_{j=1}^{m}\frac{1-d^2(x_n'-u)}{1+d(x_n-u)}\tilde{A}^{Q}(\underline{x};n) + \mathrm{uw}_A\\
&\tilde{D}^{Q}_{\bar{a}}(\underline{x};\bar{n})B_{b_m}(\underline{x}',u_m)\ldots B_{b_1}(\underline{x}',u_1)\\
&\qquad = B_{b_m}(\underline{x},u_m)\ldots B_{b_1}(\underline{x},u_1)R^{(1)}_{\bar{a}b_1}(x_n'-u_1)\ldots R^{(1)}_{\bar{a}b_m}(x_n'-u_m)\tilde{D}^{Q}_{\bar{a}}(\underline{x};\bar{n}) + \mathrm{uw}_{D_a}.
\end{aligned}$$

The 'wanted' and 'unwanted' terms have origins analogous to those for the $SU(N)$ case. If we insert these equations into the representation (3.31) of $f(\underline{x}')$ we find however, that the wanted contribution from A^Q vanishes and the wanted contribution from D^Q already gives the desired result. By definition of $\tilde{T}^Q$ and the reference state vector g we have

$$\tilde{A}^Q(\underline{x}; n)g = 0 \qquad \tilde{D}^Q_{\bar{a}}(\underline{x}; n)g = \prod_{i\in I_-} R^{(1)}_{\bar{a}i}(x_n - x_i)P_{\bar{a}n}g$$

if the index n corresponds to an antiparticle. With

$$Q^{(1)^{-1}}(\underline{x}^{(-)}, \underline{u}; n) = \mathrm{tr}_{\bar{a}}\left(\prod_{i\in I_-} R^{(1)}_{\bar{a}i}(x_n - x_i)P_{\bar{a}\bar{n}}R^{(1)}_{\bar{a}b_1}(x'_n - u_1)\ldots R^{(1)}_{\bar{a}b_m}(x'_n - u_m)\right)$$

and the assumption

$$\tilde{Q}^{-1}(\underline{x}^{(-)}, \underline{u}; n)f^{(1)}(\underline{x}^{(-)\prime}, \underline{u}) = f^{(1)}(\underline{x}^{(-)}, \underline{u})$$

it follows that the wanted term from $\tilde{D}^Q$ yields $f(\underline{x})$. Again as for the $SU(N)$ case the unwanted contributions can be written as differences which vanish after summation over the us. Because of the commutation relations of the Bs (2.12) and the symmetry property (i)$^{(1)}$ of $g^{1\ldots nb_1\ldots b_m}g(\underline{x}^{(-)}, \underline{u})$ it is sufficient to consider only the unwanted terms for $j = m$ denoted by uw^m_A and uw^m_D.

$$\mathrm{uw}^m_A(\underline{x}, \underline{u}) = d(x'_n - u_m)\tilde{B}^Q_{\bar{b}_m}(\underline{x}; m)\ldots B_{b_1}(\underline{x}, u_1)\prod_{j<m}\frac{1}{b(u_m - u_j)}A(\underline{x}, u_m)$$

$$\begin{aligned}\mathrm{uw}^m_{D_{\bar{a}}}(\underline{x}, \underline{u}) = {}&-d(x_n - u_m)\tilde{B}^Q_{\bar{a}}(\underline{x}; m)\ldots\\ &\ldots B_{b_1}(\underline{x}, u_1)\prod_{j<m}\frac{1}{b(u_j - u_m)}D_{b_m}(\underline{x}', u_m)K_{\bar{a}b_m}R^{(1)}_{b_1b_m}(u_1 - u_m)\ldots\\ &\ldots R^{(1)}_{b_{m-1}b_m}(u_{m-1} - u_m).\end{aligned}$$

With

$$\begin{aligned}D_a(\underline{x}, u)g(\underline{x}, \underline{u}) &= \prod_{i\in I_+} b(x_i - u)\prod_{i\in I_-} R^{(1)}_{ia}(x_i - u)g(\underline{x}, \underline{u})\\ Q^{(1)}(\underline{x}^{(-)}, \underline{u}; b_m) &= \mathrm{tr}_a T^{Q(1)}_a(\underline{x}^{(-)}, \underline{u}; b_m)\end{aligned}$$

(see (3.39)), the assumption (ii)$^{(1)}$, in particular the relation

$$Q^{(1)}(\underline{x}^{(-)}, \underline{u}; b_m)f^{(1)}(\underline{x}^{(-)}, \underline{u}) = f^{(1)}(\underline{x}^{(-)}, \underline{u}') \qquad (\underline{u}' = u_1, \ldots, u'_m = u_m + 2)$$

and the defining relations of the functions $\psi(x)$ and $\tau(x)$ (3.19) follows

$$\mathrm{uw}^m_A(\underline{x}, \underline{u})g(\underline{x}, \underline{u}) = -\mathrm{tr}_{\bar{a}}\mathrm{uw}^m_{D_{\bar{a}}}(\underline{x}, \underline{u}')g(\underline{x}, \underline{u}')$$

which concludes the proof.

As in section 3.1 we can iterate theorem 3.13 to get the nested generalized Bethe ansatz with levels $k = 1, \ldots, N-1$. To simplify the notation we introduce as an extension of $I_\pm$ the index sets I_k with $n_k = |I_k|$ elements for $k = 0, \ldots, N-1$ and as an extension of (3.35) and (3.37) the spaces

$$V^{(k)I_-I_k} = \bigotimes_{i\in I_-}\bar{V}^{(k)}_i \bigotimes_{i\in I_k} V^{(k)}_i \tag{3.40}$$

with basis vectors $|\bar\alpha_1 \dots \bar\alpha_{n_-} \alpha_1 \dots \alpha_{n_k}\rangle (k < \bar\alpha_i, \alpha_i \leqslant N)$. In terms of the previous notation we identify $I_+ = I_0, n_+ = n_0, V^{1\dots n} = V^{(0)I_- I_0}$ and $V^{(1)^{b_1\dots b_m}} = V^{(1)^{I_- I_1}}$. The ansatz of level k reads

$$f^{(k-1)^{I_- I_{k-1}}}(\underline{x}^{(-)}, \underline{x}^{(k-1)}) = \sum_{\underline{x}^{(k)}} B^{(k-1)}_{I_- I_{k-1}\beta_{n_k}}(\underline{x}^{(-)}, \underline{x}^{(k-1)}, x^{(k)}_{n_k}) \dots$$
$$\dots B^{(k-1)}_{I_- I_{k-1}\beta_1}(\underline{x}^{(-)}, \underline{x}^{(k-1)}, x^{(k)}_1) g^{(k-1)^{I_- I_{k-1}\beta_1\dots\beta_{n_k}}}(\underline{x}^{(-)}, \underline{x}^{(k-1)}, \underline{x}^{(k)}) \tag{3.41}$$

where for $k < N-1$ analogously to definition 3.12

$$g^{(k-1)^{I_- I_{k-1} I_k}}(\underline{x}^{(-)}, \underline{x}^{(k-1)}, \underline{x}^{(k)}) = \prod_{i=1}^{n_{k-1}} \prod_{j=1}^{n_k} \psi(x^{(k-1)}_i - x^{(k)}_j)$$
$$\times \prod_{1\leqslant i<j\leqslant n_k} \tau(x^{(k)}_i - x^{(k)}_j) f^{(k)^{I_- I_k}}(\underline{x}^{(-)}, \underline{x}^{(k)}) \otimes \Omega^{(k-1)^{I_{k-1}}}. \tag{3.42}$$

For $k = 0$ with $f^{(0)} = f$ and $\underline{x}^{(0)} = \underline{x}^{(+)}$ we have to replace $\prod_{i=1}^{n_0}$ by $\prod_{i\in I_+}$. The start of the iteration is given by a $k_{\max}$ $(1 \leqslant k_{\max} \leqslant N)$ such that all $n_k = 0$ for $k \geqslant k_{\max}$. We have to construct a vector-valued function proportional to a fixed vector which fulfils the assumptions of theorem 3.13. This is given by the following.

Lemma 3.14. The vector-valued functions for $k_{\max} < N$

$$f^{(k_{\max}-1)^{I_- I_{k_{\max}-1}}} = |\bar N \dots \bar N k_{\max} \dots k_{\max}\rangle \tag{3.43}$$

and for $k_{\max} = N$

$$f^{(N-1)^{I_- I_{N-1}}}(\underline{x}^{(-)}, \underline{x}^{(N-1)}) = \prod_{i\in I_-} \prod_{j\in I_{N-1}} \bar\psi_{N-1}(x^{(-)}_i - x^{(N-1)}_j) |\bar N \dots \bar N N \dots N\rangle \tag{3.44}$$

fulfil the conditions 3.1. The function $\bar\psi_{N-1}(x)$ has to obey the functional equation

$$(1 + d(x))\bar\psi_{N-1}(x) = \bar\psi_{N-1}(x-2) \tag{3.45}$$

which is solved by

$$\bar\psi_{N-1}(x) = \frac{\Gamma\left(\frac12 + \frac x2\right)}{\Gamma\left(\frac12 + \frac1N + \frac x2\right)}. \tag{3.46}$$

Again the general solution is obtained by multiplication with an arbitrary periodic function with period two.

Proof. Condition 3.1(i) is fulfilled because of the symmetry with respect to the particles and antiparticles among themselves. Conditions 3.1(ii) follows from the definition (3.3) of the **Q**-matrix and (2.18) of the **R**-matrix. For $k_{\max} < N$ condition 3.1(ii) follows since the **Q**-operators act as the unit operator on the state $f^{(k_{\max}-1)^{I_- I_{k_{\max}-1}}}$. For $k_{\max} = N$ we have to take into account annihilation–creation contributions of the antiparticle–particle **R**-matrix (2.18). For the last index in I_- we have

$$\tilde Q^{(N-1)I_- I_{N-1}}(\underline{x}^{(-)}, \underline{x}^{(N-1)}; n_-) = \prod_{j\in I_{N-1}} (1 + d(x^{(-)}_{n_-} + 2 - x^{(N-1)}_j))\mathbf{1}$$

and for the last index in I_{N-1}

$$Q^{(N-1)I_- I_{N-1}}(\underline{x}^{(-)}, \underline{x}^{(N-1)}; n_{N-1}) = \prod_{j\in I_-} (1 + d(x^{(-)}_j - x^{(N-1)}_{n_{N-1}}))\mathbf{1}$$

which, together with (3.45) implies the difference equations (ii) for (3.44). The solution (3.46) of (3.45) follows since $1 + d(x) = (x - 1 + 2/N)/(x-1)$.

Corollary 3.15. The system of $U(N)$ matrix difference (3.2) is solved by the generalized nested Bethe ansatz (3.41) with (3.42)–(3.45).

4. Weights of generalized Bethe vectors

In this section we analyse some group theoretical properties of generalized Bethe states. We calculate the weights of the states and show that they are highest weight states. The first result does not depend on any restriction to the states. On the other hand the second result is not only true for the conventional Bethe ansatz, which solves an eigenvalue problem and which is well known, but also, as we will show, for the generalized one which solves a difference equation (or a differential equation).

4.1. The SU(N)-case

By asymptotic expansion of the $\mathbf{R}$-matrix and the monodromy matrix T (cf (2.5) and (2.9)) we get for $u \to \infty$

$$R_{ab}(u) = \mathbf{1}_{ab} - \frac{2}{Nu} P_{ab} + \mathrm{O}(u^{-2}) \tag{4.1}$$

$$T_{1\ldots n,a}(\underline{x}, u) = \mathbf{1}_{1\ldots n,a} + \frac{2}{Nu} M_{1\ldots n,a} + \mathrm{O}(u^{-2}). \tag{4.2}$$

Explicitly we get from (2.9)

$$M_{1\ldots n,a} = P_{1a} + \cdots + P_{na} \tag{4.3}$$

where the Ps are the permutation operators. The matrix elements of $M_{1\ldots n,a}$ as a matrix in the auxiliary space are the $su(N)$ Lie algebra generators. In the following we will consider only operators acting in the fixed tensor product space $V = V^{1\ldots n}$ of (2.1); therefore we will omit the indices $1\ldots n$. In terms of matrix elements in the auxiliary space V_a the generators act on the basis states as

$$M_{\alpha}^{\alpha'} |\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n\rangle = \sum_{i=1}^{n} \delta_{\alpha'\alpha_i} |\alpha_1, \ldots, \alpha, \ldots, \alpha_n\rangle. \tag{4.4}$$

The Yang–Baxter relations (2.10) yield for $x_a \to \infty$

$$[M_a + P_{ab}, T_b(x_b)] = 0 \tag{4.5}$$

and if additionally $x_b \to \infty$

$$[M_a + P_{ab}, M_b] = 0 \tag{4.6}$$

or for the matrix elements

$$[M_{\alpha}^{\alpha'}, T_{\beta}^{\beta'}(u)] = \delta_{\alpha'\beta} T_{\alpha}^{\beta'}(u) - \delta_{\alpha\beta'} T_{\beta}^{\alpha'}(u) \tag{4.7}$$

$$[M_{\alpha}^{\alpha'}, M_{\beta}^{\beta'}] = \delta_{\alpha'\beta} M_{\alpha}^{\beta'} - \delta_{\alpha\beta'} M_{\beta}^{\alpha'}. \tag{4.8}$$

Equation (4.8) represents the structure relations of the $su(N)$ Lie algebra and (4.7) the $SU(N)$-covariance of T. In particular the transfer matrix is invariant

$$[M_{\alpha}^{\alpha'}, \mathrm{tr}T(u)] = 0. \tag{4.9}$$

We now investigate the action of the lifting operators $M_{\alpha}^{\alpha'}$ $(\alpha' > \alpha)$ to generalized Bethe vectors.

Lemma 4.1. Let $F[g](\underline{x}) \in V$ be a Bethe ansatz vector given in terms of a vector $g(\underline{x},\underline{u}) \in V^{(1)} \cong \mathbb{C}^{(N-1)\otimes m}$ by

$$F[g](\underline{x},\underline{u}) = B_{\beta_m}(\underline{x},u_m)\dots B_{\beta_1}(\underline{x},u_1)\Omega g^{\underline{\beta}}(\underline{x},\underline{u}) \tag{4.10}$$

with $\underline{\beta} = \beta_1,\dots,\beta_m$. Then $M_\alpha^{\alpha'} F[g]$ is of the form

$$M_\alpha^{\alpha'} F[g] = \begin{cases} \sum_{j+1}^{m} B_{\beta_m}\dots\delta_{\alpha'\beta_m}\dots B_{\beta_1}\Omega G_j^{\underline{\beta}}(\underline{x},\underline{u}) & \text{for } \alpha' > \alpha = 1 \\ F[M^{(1)}{}_\alpha^{\alpha'} g] & \text{for } \alpha' > \alpha > 1 \end{cases} \tag{4.11}$$

where the $M^{(1)}{}_\alpha^{\alpha'}$ are the $su(N-1)$ generators represented in $V^{(1)}$ (analogously to (4.3)) and

$$G_m(\underline{x},\underline{u}) = \left(\frac{1}{\prod_{j=1}^m b(u_m-u_j)} - \frac{\prod_{i=1}^n b(x_i-u_m)}{\prod_{j=1}^m b(u_j-u_m)} Q^{(1)}(\underline{u};m)\right) g(\underline{x},\underline{u}). \tag{4.12}$$

The operator $Q^{(1)}(\underline{u};m) \in \mathrm{End}(V^{(1)})$ is a next level Q-matrix given by the trace

$$Q^{(1)}(\underline{u};m) = \mathrm{tr}_a T_a^{Q(1)}(\underline{u};m) \tag{4.13}$$

(see (3.22)). The other G_j are obtained by Yang–Baxter relations.

Proof. First we consider the case $\alpha = 1$. The commutation rule (4.7) reads for $\beta' = 1$ and $\alpha' \to \alpha$

$$[M_1^\alpha, B_\beta(u)] = \delta_{\alpha\beta}A(u) - D_\beta^\alpha(u).$$

We commute M_1^α through all the Bs of (4.10) and use $M_1^\alpha\Omega = 0$ for $\alpha > 1$ (cf (4.4)). The As and Ds appearing are also commuted through all the Bs using the commutation rules (2.13) and (2.14). In each summand exactly one B-operator disappears. Therefore the result is of the form of (4.11). Contributions to G_m arise when we commute M_1^α through $B_{\beta_m}(u_m)$ and then push the $A(u_m)$ and $D(u_m)$ through the other Bs ($j < m$), only taking the first terms of (2.13) and (2.14) into account. All other terms would contain a $B(u_m)$ and would therefore contribute to one of the other G_j ($j < m$). Finally we apply $A(u_m)$ and $D(u_m)$ to Ω

$$A(u_m)\Omega = \Omega \qquad D_\beta^\alpha(u_m)\Omega = \delta_{\alpha\beta}\prod_{i=1}^n b(x_i-u_m)\Omega$$

and get (4.12).

For $\alpha' > \alpha > 1$ we again use the commutation rule (4.7)

$$[M_\alpha^{\alpha'}, B_\beta(u)] = \delta_{\alpha'\beta}B_\alpha(u)$$

and get

$$M_\alpha^{\alpha'} B_{\beta_m}(u_m)\dots B_{\beta_1}(u_1) = B_{\beta_m}(u_m)\dots B_{\beta_1}(u_1)M_\alpha^{\alpha'} + B_{\beta_m'}(u_m)\dots B_{\beta_1'}(u_1)M^{(1)}{}_{\underline{\beta},\alpha}^{\underline{\beta}',\alpha'}$$

with $M^{(1)}_{1\dots m,a} = P^{(1)}_{1a} + \dots + P^{(1)}_{ma}$ analogously to (4.3). Because of $M_\alpha^{\alpha'}\Omega = 0$ for $\alpha' > 1$ (cf (4.4)) we get (4.11)

The diagonal elements of M are the weight operators $W_\alpha = M_\alpha^\alpha$, they act on the basis vectors in V as

$$W_\alpha|\alpha_1,\dots,\alpha_n\rangle = \sum_{i=1}^n \delta_{\alpha_i\alpha}|\alpha_1,\dots,\alpha_n\rangle \tag{4.14}$$

which follows from $P_{i\alpha}^{\alpha'}|\alpha_i\rangle = \delta_{\alpha\alpha_i}|\alpha'\rangle$. In particular we get for the Bethe ansatz reference state (3.14)

$$W_\alpha \Omega = \delta_{\alpha 1} n\Omega. \tag{4.15}$$

Lemma 4.2. Let $F[g] \in V^{1\ldots n}$ be as in lemma 4.1. Then

$$W_\alpha F[g] = \begin{cases} (n-m)F[g] & \text{for } \alpha = 1 \\ F[W_\alpha^{(1)} g] & \text{for } \alpha > 1 \end{cases} \tag{4.16}$$

where the $W_\alpha^{(1)}$s are the $su(N-1)$ weight operators acting in $V^{(1)}$, i.e. the diagonal elements of generator matrix $M^{(1)}{}_\alpha^{\alpha'}$ (analogously to (4.3)).

Proof. By means of the commutation relation (4.7) for $\alpha' = \alpha = \beta' = 1$, $\beta > 1$

$$[W_1, B_\beta] = -B_\beta$$

we commute W_1 through all m Bs of (4.10) and with (4.15) we get the first equation. For the second equation we again use (4.7) now for $\alpha' = \alpha > 1$, $\beta' = 1$, $\beta > 1$

$$[W_\alpha, B_\beta] = \delta_{\alpha\beta} B_\beta.$$

Again commuting W_α through all the Bs of (4.10) we get with (4.15)

$$\begin{aligned} W_\alpha B_{\beta_m} \ldots B_{\beta_1} \Omega g^{\beta_1\ldots\beta_m} &= B_{\beta_m} \ldots B_{\beta_1} \left(W_\alpha + \sum_{i=1}^m \delta_{\beta_i \alpha} \right) \Omega g^{\beta_1\ldots\beta_m} \\ &= B_{\beta_m} \ldots B_{\beta_1} \Omega (W_\alpha^{(1)} g)^{\beta_1\ldots\beta_m} \end{aligned}$$

which concludes the proof.

Theorem 4.3. Let the vector-valued function $f(\underline{x}) \in V$ be given by the Bethe ansatz 3.5 fulfilling the assumptions of theorem 3.7. If in addition $f^{(1)}$ is a highest weight vector and an eigenvector of the weight operators with

$$W_\alpha^{(1)} f^{(1)} = w_\alpha^{(1)} f^{(1)} \tag{4.17}$$

then also f is a highest weight vector

$$M_\alpha^{\alpha'} f = 0 \qquad (\alpha' > \alpha) \tag{4.18}$$

and an eigenvector of the weight operators

$$W_\alpha f = w_\alpha f \qquad w_\alpha = \begin{cases} n - m & \text{for } \alpha = 1 \\ w_\alpha^{(1)} & \text{for } \alpha > 1 \end{cases} \tag{4.19}$$

with

$$w_\alpha \geqslant w_\beta \qquad (1 \leqslant \alpha < \beta \leqslant N). \tag{4.20}$$

Proof. To prove the highest weight property we apply lemma 4.1. By assumption $f^{(1)}$ fulfils the difference equation

$$f^{(1)}(u_1, \ldots, u_m + 2) = Q^{(1)}(\underline{u}; m) f^{(1)}(u_1, \ldots, u_m).$$

Together with (3.18), (3.19) and (4.12) we obtain after summation $\sum_{u_m} G_m(\underline{x}, \underline{u}) = 0$, if $u_m = \tilde{u}_m - 2l_m (l_m \in \mathbb{Z})$. The same is true for the other G_i in (4.11), since g fulfils the symmetry property of condition 3.1(i) and thereby $F[g](\underline{x}, \underline{u})$ of (4.10) is symmetric with respect to the u_i. Therefore in (4.12) we have $M_\alpha^{\alpha'} f = 0$ for $\alpha' > \alpha > 1$ and for $\alpha' > \alpha = 1$

by assumption on $f^{(1)}$. The weights of f follow from lemma 4.2 and also by assumption on $f^{(1)}$. From the commutation rule (4.8) and $M_\alpha^{\beta\dagger} = M_\beta^\alpha$ follows

$$0 \leqslant M_\alpha^\beta M_\beta^\alpha = M_\beta^\alpha M_\alpha^\beta + W_\alpha - W_\beta$$

which implies (4.20).

Since the states $f^{(k_{\max}-1)}$ of (3.25) are highest weight states in $V^{(k_{\max}-1)}$ with weight $w_{k_{\max}}^{(k_{\max}-1)} = n_{k_{\max}-1}$ we have the following.

Corollary 4.4. If $f(\underline{x})$ is a solution of the system of $SU(N)$ matrix difference equations (3.2)

$$f(\ldots, x_i + 2, \ldots) = Q(\underline{x}; i) f(\ldots, x_i, \ldots) \qquad (i = 1, \ldots, n)$$

given by the generalized nested Bethe ansatz of corollary 3.9, then f is a highest weight vector with weights

$$w = (w_1, \ldots, w_N) = (n - n_1, n_1 - n_2, \ldots, n_{N-2} - n_{N-1}, n_{N-1}) \tag{4.21}$$

where n_k is the number of $B^{(k-1)}$ operators in the Bethe ansatz of level k, $(k = 1, \ldots, N-1)$. Further non-highest weight solutions of (3.2) are given by

$$f_\alpha^{\alpha'} = M_\alpha^{\alpha'} f \qquad (\alpha' < \alpha). \tag{4.22}$$

The interpretation of (4.21) is that each $B^{(k)}$-operator reduces w_k and lifts a w_l $(l > k)$ by one.

4.2. The $U(N)$-case

The results of this section and also the techniques used are very similar to the corresponding ones of the previous section. Therefore we only point out the main differences. By an asymptotic expansion of the **R**-matrix and the monodromy matrix T (cf (2.5), (2.18), (2.20) and (2.23)) we get for $u \to \infty$

$$R_{ab}(u) = \mathbf{1}_{ab} - \frac{2}{Nu} P_{ab} + \mathrm{O}(u^{-2}) \tag{4.23}$$

$$R_{\bar{a}b}(u) = \mathbf{1}_{ab} + \frac{2}{Nu} K_{\bar{a}b} + \mathrm{O}(u^{-2}) \tag{4.24}$$

$$T_{1\ldots n,a}(\underline{x}, u) = \mathbf{1}_{1\ldots n,a} + \frac{2}{Nu} M_{1\ldots n,a} + \mathrm{O}(u^{-2}). \tag{4.25}$$

Explicitly we get from (2.23)

$$M_{1\ldots n,a} = \sum_{i\in I_+} P_{ia} - \sum_{i\in I_-} K_{ia} \tag{4.26}$$

where $I_\pm$ denote the particles and antiparticles, respectively.

In the following we will suppress the indices like $1\ldots n$. In terms of matrix elements in the auxiliary space V_a the generators act on the basis states as

$$M_\alpha^{\alpha'} |\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n\rangle = \left(\sum_{i\in I_+} \delta_{\alpha'\alpha_i} - \sum_{i\in I_-} \delta_{\alpha'\alpha_i}\right) |\alpha_1, \ldots, \alpha, \ldots, \alpha_n\rangle. \tag{4.27}$$

The commutation relations of the M and T which follow from the Yang–Baxter relations are the same as in section 4.1. Also lemma 4.1 holds here for the $U(N)$ case. However, one has to replace Ωg in (4.10) by g as in Bethe ansatz 3.11, and the **Q**-matrix here is given by the trace of (3.39).

The slight difference to the $SU(N)$ case arises from the action of the Cartan sub-algebra; i.e. from the diagonal elements of M which are the weight operators $W_\alpha = M_\alpha^\alpha$. They act on the basis vectors in V as

$$W_\alpha |\alpha_1, \ldots, \alpha_n\rangle = \left(\sum_{i \in I_+} \delta_{\alpha\alpha_i} - \sum_{i \in I_-} \delta_{\alpha\alpha_i} \right) |\alpha_1, \ldots, \alpha_n\rangle \tag{4.28}$$

which follows from (4.27). In particular we have for the Bethe ansatz reference state g

$$W_1 g = n_+ g. \tag{4.29}$$

This means that also lemma 4.2 holds here for the $U(N)$ case where, however, in (4.16) the number n is replaced by n_+.

Theorem 4.5. Let the vector-valued function $f(\underline{x}) \in V$ be given by the Bethe ansatz 3.11 fulfilling the conditions of theorem 3.13, i.e. (3.38) and (3.19). If in addition $f^{(1)}$ is a highest weight vector and an eigenvector of the weight operators with

$$W_\alpha^{(1)} f^{(1)} = w_\alpha^{(1)} f^{(1)} \tag{4.30}$$

then f is also a highest weight vector

$$M_\alpha^{\alpha'} f = 0 \qquad (\alpha' > \alpha) \tag{4.31}$$

and an eigenvector of the weight operators

$$W_\alpha f = w_\alpha f \qquad w_\alpha = \begin{cases} n_+ - m & \text{for } \alpha = 1 \\ w_\alpha^{(1)} & \text{for } \alpha > 1 \end{cases} \tag{4.32}$$

with

$$w_\alpha \geqslant w_\beta \qquad (1 \leqslant \alpha < \beta \leqslant N). \tag{4.33}$$

The proof of this theorem is again parallel to the corresponding one section 4.1.

The states $f^{(k_{\max}-1)}$ of lemma 3.14 which define the start of the iteration of the nested Bethe ansatz are obviously highest weight states in $V^{(k_{\max}-1)}$ with weight $w_{k_{\max}}^{(k_{\max}-1)} = n_{k_{\max}-1} - n_-$ by (4.28).

Corollary 4.6. If $f(\underline{x})$ is a solution of the system of $U(N)$ matrix difference equations (3.3)

$$f(\ldots, x_i + 2, \ldots) = Q(\underline{x}; i) f(\ldots, x_i, \ldots) \qquad (i = 1, \ldots, n)$$

given by the generalized nested Bethe ansatz of corollary 3.15, then f is a highest weight vector with weights

$$w = (w_1, \ldots, w_N) = (n_+ - n_1, n_1 - n_2, \ldots, n_{N-2} - n_{N-1}, n_{N-1} - n_-) \tag{4.34}$$

where n_k is the number of $B^{(k)}$ operators in the Bethe ansatz of level k. Further non-highest weight solutions of (3.2) are given by

$$f_\alpha^{\alpha'} = M_\alpha^{\alpha'} f \qquad (\alpha' < \alpha). \tag{4.35}$$

Note that in contrast to the $SU(N)$ case, here the weights may also be negative.

5. Examples

5.1. The SU(N)-case

From a solution of the matrix difference equations (3.2) one gets a new solution by multiplication of a scalar function which is symmetric with respect to all variables x_i and periodic with period two. Therefore the solutions of the following examples may be multiplied by such functions.

Example 5.1. The simplest example is obtained for $k_{\max} = 1$ which means the trivial solution of the difference equations

$$f^{1\dots n} = \Omega^{1\dots n}.$$

The weights of $f^{1\dots n}$ are $w = (n, 0, \dots, 0)$.

In the language of spin chains this case corresponds to the ferromagnetic groundstate.

Example 5.2. For the case $k_{\max} = 2$ and $n^{(1)} = 1$ the solution reads

$$f^{1\dots n}(\underline{x}) = \sum_u B_{1\dots n,\beta}(\underline{x}, u)\Omega^{1\dots n} g^{\beta}(\underline{x}, u)$$

with $u = \tilde{u} - 2l$ ($l \in \mathbb{Z}$, $\tilde{u}$ an arbitrary constant) and

$$g^{\beta}(\underline{x}, u) = \delta_{\beta 2} \prod_{i=1}^{n} \psi(x_i - u).$$

The weights of this vector $f^{1\dots n}$ are $w = (n - 1, 1, 0, \dots, 0)$. The action of the creation operator $B_{1\dots n,\beta}(x, y; u)$ on the reference state is easily calculated with help of (2.6), (2.9) and (2.11).

As a particular case of this example we determine explicitly the solution for the following.

Example 5.3. The action of the B-operator on the reference state for the case of $n = 2$ of example 5.2 yields

$$B_{12,\beta}(x, y; u)|11\rangle = c(x - u)b(y - u)|\beta 1\rangle + c(y - u)|1\beta\rangle.$$

Therefore we obtain

$$f^{12}(x, y) = \sum_u \psi(x - u)\psi(y - u)\{c(x - u)b(y - u)|21\rangle + c(y - u)|12\rangle\}$$

with $u = \tilde{u} - 2l$, ($l \in \mathbb{Z}$). Using the expressions for the functions b, c, ψ given by (2.7) and (3.20) we get (by Dougall's formula) up to a constant

$$f^{12}(x, y) = \left(\sin\pi\left(\frac{x - \tilde{u}}{2} - \frac{1}{N}\right)\sin\pi\left(\frac{y - \tilde{u}}{2} - \frac{1}{N}\right)\Gamma\left(\frac{y - x}{2} + \frac{1}{N}\right)\right.$$
$$\left.\times\Gamma\left(1 + \frac{x - y}{2} + \frac{1}{N}\right)\right)^{-1}(|21\rangle - |12\rangle).$$

This solution could also be obtained by means of the method used in [3], namely by diagonalization of the $\mathbf{R}$-matrix. One obtains the difference equations

$$f_-(x, y) = R_-(x - y) f_-(y, x) \qquad f_-(x, y) = f_-(y, x + 2)$$

with the eigenvalue $R_-(x) = (x + 2/N)/(x - 2/N)$ of the antisymmetric tensor representation.

Example 5.4. Next we consider for $N > 2$ the case of the quantum space $V_{123} = V_1 \otimes V_2 \otimes V_3$ and the case that the nested Bethe ansatz has only two levels with two creation operators in the first level and one in the second level. This means $k_{\max} = 3$, $n = 3$, $n^{(1)} = 2$, $n^{(2)} = 1$ and the weights $w = (1, 1, 1, 0, \ldots, 0)$. The first level Bethe ansatz is given by

$$f^{123}(x, y, z) = \sum_{u,v} B_{123,\beta}(x, y, z; v) B_{123,\alpha}(x, y, z; u) \Omega^{123} g^{\alpha\beta}(x, y, z; u, v)$$

where the summation is specified by $u = \tilde{u} - 2k$, $v = \tilde{u} - 2l$, $(k, l \in \mathbb{Z})$. By (3.18) g^{12} is related to the next level function $f^{(1)12}$ by

$$g^{12}(x, y, z; u, v) = \prod_{x_i = x,y,z} \prod_{u_j = u,v} \psi(x_i - u_j) \tau(u - v) f^{(1)12}(u, v).$$

The second level Bethe ansatz reads

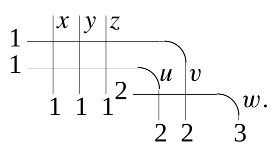
$$f^{(1)12}(u, v) = \sum_{w} B^{(1)}_{12\gamma}(u, v; w) \Omega^{(1)12} g^{(1)\gamma}(u, v; w)$$

where $w = \tilde{w} - 2m$, $(m \in \mathbb{Z})$. The second level reference state is $\Omega^{(1)12} = |22\rangle^{(1)} \in V^{(1)12}$. Again according to (3.18)

$$g^{(1)\gamma}(u, v; w) = \psi(u - w)\psi(v - w) f^{(2)\gamma}$$

with $f^{(2)\gamma} = \delta_{\gamma 3}$. As in example 5.3 the action of the operators B and $B^{(1)}$ on their reference states may be calculated.

For this example the two-level nested Bethe ansatz may be depicted as



5.2. The U(N)-case

Example 5.5. Let us consider the trivial case that there is no B-operator in each level of the nested Bethe ansatz, which means that $k_{\max}$ of section 3 is equal to one. In the language of the conventional Bethe ansatz for quantum chains this corresponds to the 'ferromagnetic vacuum'. By section 4 this means that $f^{1\ldots n}(\underline{x})$ has the weights

$$w = (n_+, 0, \ldots, 0, -n_-).$$

For fixed positions of the particles I_+ and antiparticles I_- by lemma 3.14 the vector $f^{1\ldots n} \in V^{I_- I_+}$ is given by

$$f^{\alpha_1\ldots\alpha_n}(\underline{x}) = \prod_{i \in I_-} \delta_{\alpha_i \bar{N}} \prod_{i \in I_+} \delta_{\alpha_i 1}$$

or if the antiparticles are sitting at the first places

$$f^{1\ldots n} = |\bar{N} \ldots \bar{N} 1 \ldots 1\rangle.$$

This $f^{1\ldots n}$ is a highest weight vector in V_{ref}.

Example 5.6. For $N > 2$ and $n_+ \geqslant 2$ let us take the case where there is one B-operator in the first level of the nested Bethe ansatz and no Bs in higher levels, which means that the $k_{\max}$ of section 3 is equal to two. By section 4 this means that the weights are

$$w = (n_+ - 1, 1, 0, \dots, 0, -n_-).$$

The first level ansatz reads

$$f^{1\dots n}(\underline{x}) = \sum_u B_{1\dots n\beta}(\underline{x}; u) g^{1\dots n\beta}(\underline{x}; u). \tag{5.1}$$

For fixed positions of the particles I_+ and antiparticles I_-, if the antiparticles are sitting on the left of all particles, the function $g^{1\dots n\beta}$ is given by

$$g^{I_- I_+ \beta}(\underline{x}, u) = \prod_{i \in I_+} \psi(x_i - u) f^{(1)I_- \beta} \otimes \Omega^{I_+} \tag{5.2}$$

with $\psi(x) = \Gamma\left(1 - \frac{1}{N} + \frac{x}{2}\right) / \Gamma\left(1 + \frac{x}{2}\right)$ (see (3.19)) and by lemma 3.14

$$f^{(1)I_- I_1} = |\bar{N} \dots \bar{N} 2\rangle \qquad \Omega^{I_+} = |1 \dots 1\rangle \tag{5.3}$$

i.e. $f^{(1)}$ is the highest weight vector in $V^{(1)}_{\mathrm{ref}}$. Note that the function ψ appears only with respect to the parameter x_i which correspond to the particles. The action of the B-operators in (5.1) can easily be obtained from the definition of the $\mathbf{R}$-matrices (2.17) and (2.18). In particular we consider the following.

Example 5.7. As a simple case of example 5.6 we take $n_- = 1$ and $n_+ = 2$, which means $w = (1, 1, 0, \dots, 0, -1)$

$$f^{\bar{1}23}(x, y, z) = \sum_u \psi(y - u)\psi(z - u)\{c(y - u)b(z - u)|\bar{N}21\rangle + c(z - u)|\bar{N}12\rangle\}.$$

The sum over u can be performed and gives the same result as in example 5.3.

Example 5.8. For $N = 2$ let $n_- = n_+$ be $= 1$. In addition to the trivial case of example 5.5 with no B-operator ($w = (1, -1)$) there is only the possibility analogous to examples 5.6 and 5.7 with one B-operator. By section 4 $f^{\bar{1}2}(x, y)$ is an $U(2)$-singlet vector with the weights $w = (0, 0)$. For $N = 2$ we must take into account the annihilation–creation contribution, for the action of the B-operator on the reference state as well as for $f^{(1)}$ due to lemma 3.14:

$$f^{\bar{1}2}(x, y) = \sum_u \bar{\psi}_1(x - u)\psi(y - u)\{d(x - u)b(y - u)|\bar{1}1\rangle + c(y - u)|\bar{2}2\rangle\}$$

with $\psi(x) = \bar{\psi}_1(x) = \Gamma\left(\frac{1}{2} + \frac{x}{2}\right) / \Gamma\left(1 + \frac{x}{2}\right)$. As in example 5.7 the sum over $u = \tilde{u} - 2l$, ($l \in \mathbb{Z}$) can be performed and we get for any $\tilde{u}$ the solution

$$f^{\bar{1}2}(x, y) = \frac{\cos\frac{\pi}{2}(x - y)}{\cos\frac{\pi}{2}(x - \tilde{u})\cos\frac{\pi}{2}(y - \tilde{u})} \frac{1}{x - y + 1}\{|\bar{1}1\rangle + |\bar{2}2\rangle\}.$$

As a generalization of this formula for arbitrary N we consider the following.

Example 5.9. Let us take for $n_- = n_+ = 1$ the case where there is exactly one B-operator in each level of the nested Bethe ansatz, which means that the $k_{\max}$ of section 3 is equal to $N - 1$. By section 4 this means the weights are $w = (0, \dots, 0)$, i.e. $f^{12}(x, y)$ is an $U(N)$-singlet. The first level ansatz reads

$$f^{\bar{1}2}(x, y) = \sum_u B_{\bar{1}2,\beta}(x, y; u) g^{\bar{1}2\beta}(x, y; u) \tag{5.4}$$

where $u = \tilde{u} + 2l$, $l \in \mathbb{Z}$ and

$$g^{\bar{1}20}(x, y, u) = \psi(y-u) f^{(1)^{\bar{1}0}}(x, u)\Omega^2. \tag{5.5}$$

The higher-level ansatze are of the same form. A particular solution is

$$f^{\bar{\alpha}\beta}(x, y) = \frac{1}{x - y + 1}\delta_{\alpha\beta}. \tag{5.6}$$

Proof. The solution (5.6) is, up to a constant, the particular case $k = 0$ of a general formula valid for all levels. This general formula (again up to unimportant constants)

$$f^{(k)^{\bar{\alpha}\beta}}(x, y) = \bar{\psi}_k(x-y)\delta_{\alpha\beta} \qquad (k < \alpha, \beta \leqslant N) \tag{5.7}$$

where as an extension of (3.46)

$$\bar{\psi}_k(x) = \frac{\Gamma\left(\frac{1}{2} + \frac{x}{2}\right)}{\Gamma\left(\frac{3}{2} - \frac{k}{N} + \frac{x}{2}\right)} \tag{5.8}$$

will be proved inductively. For $k = N - 1$ as the start of the iteration formula (5.7) follows from lemma 3.14. For the other values of k it follows recursively from (3.41) and (3.42):

$$f^{(k-1)^{\bar{1}2}}(x, y) = \sum_u B^{(k-1)}_{\bar{1}2\beta}(x, y; u) g^{(k-1)^{\bar{1}2\beta}}(x, y; u) \tag{5.9}$$

$$\sum_u B^{(k-1)}_{\bar{1}2\beta}(x, y; u)|\bar{\alpha}k\rangle = d(x-u)b(y-u)|\bar{k}k\rangle + c(y-u)|\bar{\alpha}\beta\rangle \tag{5.10}$$

$$g^{(k-1)^{\bar{\alpha}\beta\gamma}}(x, y; u) = \psi(y-u) f^{(k)^{\bar{\alpha}\gamma}}(x, u)\delta_{\beta k}. \tag{5.11}$$

We calculate the right-hand side of (5.9) inserting (5.10) and (5.11) with $f^{(k)}$ given by (5.7) and with $u = x + 2l$

$$\begin{aligned}
&\sum_l \psi(x-u)\bar{\psi}_k(y-x)\left\{d(x-u)b(y-u)(N-k)|\bar{k}k\rangle + c(y-u)\sum_{\alpha=k+1}^{N}|\bar{\alpha}\alpha\rangle\right\} \\
&\quad = \frac{1}{N}\sum_l \left\{\frac{\Gamma\left(-\frac{1}{N}+l\right)}{\Gamma(l)} \frac{\Gamma\left(-\frac{1}{2}+\frac{x-y}{2}+l\right)}{\Gamma\left(\frac{3}{2}-\frac{k}{N}+\frac{x-y}{2}+l\right)}(N-k)|\bar{k}k\rangle\right. \\
&\qquad \left. + \frac{\Gamma\left(-\frac{1}{N}+l\right)}{\Gamma(1+l)} \frac{\Gamma\left(\frac{1}{2}+\frac{x-y}{2}+l\right)}{\Gamma\left(\frac{3}{2}-\frac{k}{N}+\frac{x-y}{2}+l\right)} \sum_{\alpha=k+1}^{N}|\bar{\alpha}\alpha\rangle\right\} \\
&\quad = \text{constant } \bar{\psi}_{k-1}(x-y)\sum_{\alpha=k}^{N}|\bar{\alpha}\alpha\rangle = \text{constant } f^{(k-1)^{\bar{1}2}}(x, y).
\end{aligned} \tag{5.12}$$

The sums over l were performed using the Gauss formula

$$\sum_l \frac{\Gamma(a+l)\Gamma(b+l)}{l!\Gamma(c+l)} = \frac{\Gamma(c-a-b)\Gamma(a)\Gamma(b)}{\Gamma(c-a)\Gamma(c-b)}.$$

This concludes the proof of formula (5.7).

Example 5.10. The last example will be used in [17] to calculate the exact three-particle form factor of the fundamental field in the chiral Gross–Neveu model. We take $n_- = 1$, $n_+ = 2$ and again exactly one B-operator in each level of the nested Bethe ansatz. By section 4 this means that $f^{\bar{1}23}(x, y, z)$ is a $U(N)$-vector with weights

$$w = (1, 0, \ldots, 0).$$

The first level ansatz reads

$$f^{\bar{1}23}(x, y, z) = \sum_u B_{\bar{1}23,\beta}(x, y, z; u) g^{\bar{1}23\beta}(x, y, z; u). \tag{5.13}$$

The higher level ($k > 1$) Bethe ansatz coincides with that of example 5.9. A solution of the difference equations for this case is given by

$$\begin{aligned} f^{\bar{1}23}(x, y, z) = \sum_u \Bigg\{ & \frac{\Gamma\left(-\frac{1}{N} + \frac{x-u}{2}\right)}{\Gamma\left(\frac{x-u}{2}\right)} \frac{\Gamma\left(-\frac{1}{N} + \frac{y-u}{2}\right)}{\Gamma\left(\frac{y-u}{2}\right)} \frac{\Gamma\left(-\frac{1}{2} + \frac{z-u}{2}\right)}{\Gamma\left(\frac{3}{2} - \frac{1}{N} + \frac{z-u}{2}\right)} (N-1)|\bar{1}11\rangle \\ & - \frac{\Gamma\left(-\frac{1}{N} + \frac{x-u}{2}\right)}{\Gamma\left(\frac{x-u}{2}\right)} \frac{\Gamma\left(-\frac{1}{N} + \frac{y-u}{2}\right)}{\Gamma\left(1 + \frac{y-u}{2}\right)} \frac{\Gamma\left(\frac{1}{2} + \frac{z-u}{2}\right)}{\Gamma\left(\frac{3}{2} - \frac{1}{N} + \frac{z-u}{2}\right)} \sum_{\alpha=2}^{N} |\bar{\alpha}\alpha 1\rangle \\ & - \frac{\Gamma\left(-\frac{1}{N} + \frac{x-u}{2}\right)}{\Gamma\left(1 + \frac{x-u}{2}\right)} \frac{\Gamma\left(1 - \frac{1}{N} + \frac{y-u}{2}\right)}{\Gamma\left(1 + \frac{y-u}{2}\right)} \frac{\Gamma\left(\frac{1}{2} + \frac{z-u}{2}\right)}{\Gamma\left(\frac{3}{2} - \frac{1}{N} + \frac{z-u}{2}\right)} \sum_{\alpha=2}^{N} |\bar{\alpha}1\alpha\rangle \Bigg\} \end{aligned} \tag{5.14}$$

where $u = \tilde{u} + 2l$, $l \in \mathbb{Z}$.

Proof. Analogously to (5.10) and (5.11) we have for ($k = 1$)

$$\begin{aligned} B_{\bar{1}23,\beta}(x, y, z; u)|\bar{\alpha}11\rangle &= d(x-u)b(y-u)b(z-u)\delta_{\alpha\beta}|\bar{1}11\rangle + c(y-u)b(z-u)|\bar{\alpha}\beta 1\rangle \\ &\quad + c(z-u)|\bar{\alpha}1\beta\rangle \\ g^{\bar{\alpha}\beta}(x, y, z; u) &= \psi(y-u)\psi(z-u) f^{(1)\bar{\alpha}\beta}(x, u) \end{aligned}$$

where $f^{(1)\bar{\alpha}\beta}(x, u)$ is the same function as in example 5.9 and given by (5.7) for $k = 1$. Inserting this into (5.13) we get by analogy to (5.12) the result (5.14).

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